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1987 J. Phys. A: Math. Gen. 20 L213

(http://iopscience.iop.org/0305-4470/20/4/006)

J. Phys. A: Math. Gen. **20** (1987) L213–L216. Printed in the UK

LETTER TO THE EDITOR

Characteristics of Suzuki's indirect fluctuating mean-field approximation

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Received 11 November 1986

Abstract. Results based on the indirect fluctuating mean-field theory recently proposed by Suzuki are reported. The results indicate some of the important characteristics of this approximation method, which differs in significant ways from the standard mean-field approximation and recent improvements of it.

Methods of approximating the critical temperature and other properties of statistical mechanical systems have been of interest for many years. For the Ising model, such approximations stretch back to the molecular or mean-field theory of Bragg and Williams (1934) and continue today (see, for example, Burley 1972, Muller-Hartmann and Zittartz 1977, Plascak and Silva 1982). In one recent very interesting paper by Suzuki (1986a) two variations of the mean-field method were made. Suzuki labels these as the direct fluctuating mean-field (DFMF) and the indirect fluctuating mean-field (IFMF) approximations.

For lattice spin systems with spin variables $\sigma = \pm 1$ and interaction strength J, the DFMF rather than replacing interactions of the form $(J/KT)\sigma_i\sigma_j$ with $(J/KT)m\sigma_j$, as in the usual method, replaces them with $\tanh(J/KT)m\sigma_j$, where m is the mean-field magnetisation, K is Boltzmann's constant and T is the temperature. This attempts to take into account the direct fluctuation between σ_i and σ_j. Suzuki points out that the spin at the jth site feels directly the fluctuating mean field in his above method and this makes the fluctuation effects due to the whole system weak. The IFMF as the name implies stresses the indirect effects. It does so by simply eliminating the direct ones. That is, considering $\langle\sigma_i\rangle$, the thermal average of σ_i, of an arbitrary cluster of sites, as in figure 1, all the fluctuating mean fields on the ith site are excluded in the calculation of $\langle\sigma_i\rangle$, while any fluctuating mean field at the other sites of the cluster are included. In both methods the self-consistency condition $\langle\sigma_i\rangle = m$ is used. For the isotropic nearest-neighbour Ising model on a square lattice Suzuki (1986a), using a four-site cluster (see figure 2(a)), obtains a critical temperature of $J/KT_c = 0.4291$ through the IFMF approximation while one has the exact result $J/KT_c = 0.4407$ by Onsager (1944). The method thus gives a very good result compared to other approximations using a small-sized cluster. For comparison, if one replaces all interactions on the boundary of the cluster with $Jm\sigma_i$, $i = 1$, 2, 3 or 4, one only obtains $J/KT_c = 0.2857$. Also, by looking at a two-site cluster Suzuki (1986a) shows that one has $T_c = 0$ for the one-dimensional nearest-neighbour Ising model using the IFMF method.

Due to the abovementioned successes, it is useful to see results based on other cluster sizes and for other lattice systems so that one understands the characteristics

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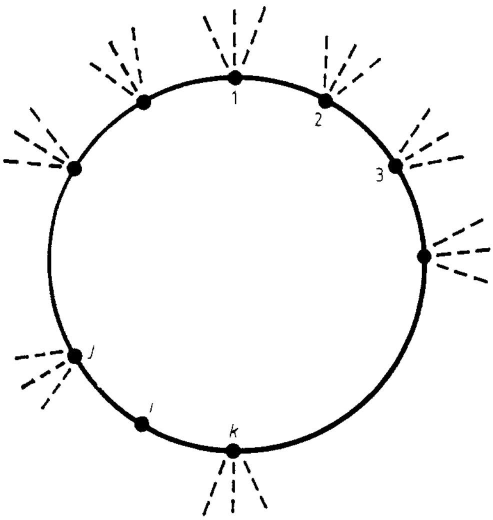

Figure 1. A general cluster showing boundary sites with broken lines indicating the $\tanh(J)m$ factors of the DFMF theory and no such factors for the ith site, in accordance with the IFMF theory.

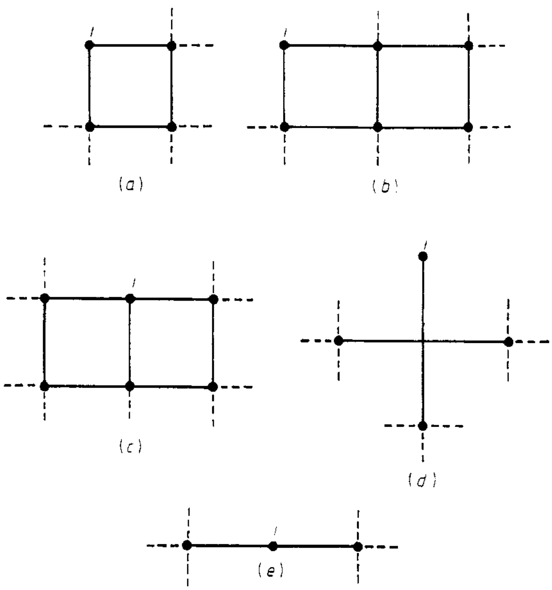

Figure 2. (*a*) A four-site cluster; (*b*) and (*c*) six-site clusters; (*d*) five-site cluster and (*e*) three-site cluster; all for the square lattice.

of the method better. We first examine the method applied to other two-dimensional lattices where exact results are available for comparison; then we look at the effect of increasing the cluster size and we finish by showing some effects of cluster shape and size.

For nearest-neighbour interactions, the triangle lattice and the honeycomb or hexagonal lattice have exactly determined critical temperatures and are therefore natural choices with which to compare IFMF estimates. For the triangle lattice, using a three-site cluster forming an elementary triangle one obtains from the IFMF method

$$\langle\sigma_i\rangle = \frac{x^3 \sinh(8mT) - 2(1/x)\sinh(4mT)}{x^3\cosh(8mT) + 4(1/x) + (2/x)\cosh(4mT)} \tag{1}$$

where $T = \tanh(\beta J)$, $x = \exp(\beta J)$ and $\beta = 1/KT$. Requiring that $\langle\sigma_i\rangle = m$ and looking for T_c such that $m \to 0$ as $T \to T_c$ from above, one obtains $\beta_c J = 0.3241$,while the exact value is $\beta_c J = 0.2747$. Looking at the same limit and using a six-site cluster for the hexagonal lattice with the six sites forming a hexagon, the IFMF method calculation gives $\beta_c J = 0.5609$ whereas the exact result is $\beta_c J = 0.6585$. One very important characteristic becomes apparent when looking at these two results. In one case, that of the hexagonal lattice, we have an estimated value of $\beta_c J$ which is too low. This is the usual case with the standard mean-field results, Bethe approximation, Kikuchi approximations, etc. However, in the case of the triangular lattice we have the opposite situation where the value of $\beta_c J$ is too large. This is counter to the usual mean-field, Bethe, Kikuchi and other systematic cluster approximations which have been proved (see Griffiths 1967, Krinsky 1975, Vigfusson 1985, Monroe 1985, 1986) to give upper bounds on T_c, i.e. lower bounds on $\beta_c J$.

When looking at larger clusters, one also has some new properties by using this method. Considering a six-site cluster, as shown in figure 2(*b*), one finds $\beta_c J = 0.4207$. However, if one uses the six-site cluster of figure 2(*c*) one finds $\beta_c J = 0.3576$. Therefore in either case, going to the larger cluster actually worsens the approximation rather than bettering it. This is in contrast to the usual result. In the case of the systematic cluster mean-field approach or the DFMF approach one would retain the external interactions $(J/KT)\sigma_i m$ or $\tanh(J/KT)m\sigma_i$, respectively, on the *i*th site. Then for the four-site cluster one would obtain as an approximation for the critical temperature $\beta_c J = 0.2857$ for the four-site cluster mean field and $\beta_c J = 0.2910$ for the DFMF approach. Going to the six-site cluster and using DFMF on the cluster of figure 2(*b*) one obtains an estimate of $\beta_c J = 0.2974$ or, if the cluster of figure 2(*c*) is used, $\beta_c J = 0.3065$. Both are improvements over what is obtained in the four-site cluster approximation using DFMF. Similar sorts of improvements would occur using a cluster mean-field approach. This improvement with increasing size of the cluster has been exploited by various authors (Vigfusson 1985, Suzuki 1986b, c) but is not a characteristic of the IFMF approach.

Finally, one can show that the approximation is very dependent on the shape of the cluster. Using the five-site cluster shown in figure 2(*d*) one obtains $\beta_c J = 0.5240$ from the IFMF approach which is, as was the case for $\beta_c J$ for the triangle lattice, too large. However, if one goes to only a three-site cluster as shown in figure 2(*e*) one obtains $\beta_c J = 0.4335$ which is a very good approximation to the exact result. In fact, it is the best result of any of the IFMF approximations for the nearest-neighbour square lattice Ising model, while at the same time the smallest cluster analysed. If one goes to a two-site cluster one, however, does not improve upon this result but rather drastically overestimates the value of $\beta_c J$, the IFMF approach giving $\beta_c J = 0.6115$.

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